The Strong Consistency of Neutral and Monotonic Binary Social Decision Rules

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Abstract

The purpose of this paper is to investigate the strong consistency of neutral and monotonic binary social decision rules. Individuals are assumed to satisfy von Neumann - Morgenstern axioms of individual rationality. The main result of the paper shows that there does not exist any neutral and monotonic non-null non-dictatorial binary social decision rule which is strongly consistent. The relationship between restricted preferences and the existence of strong equilibria is also investigated. It is shown that for every non-dictatorial social decision function satisfying the conditions of independence of irrelevant alternatives, neutrality, monotonicity and weak Pareto-criterion there exists a profile of individual orderings satisfying value-restriction corresponding to which there is no strong equilibrium.

Key Words: Binary Social Decision Rules, Strong Consistency, Neutrality, Monotonicity, Value-Restricted Preferences

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1 Introduction

The purpose of this paper is to investigate the strong consistency of neutral and monotonic binary social decision rules. A social decision rule is defined to be strongly consistent iff corresponding to every profile of individual orderings over the set of outcomes there is a strong equilibrium. The notion of strong consistency, introduced by Peleg [1978, 1984], has been extensively discussed in the context of social choice functions [see Moulin (1983) and Peleg (1984) among others]. The question of strong consistency of Arrowian social decision rules, however, has received relatively less attention, particularly under the assumption that individuals satisfy von Neumann - Morgenstern axioms of individual rationality.

Let S be the finite set of social alternatives and N the finite set of individuals. Let $\#N = n$. Let $\mathcal{C}$ denote the set of reflexive and connected binary relations on S, $\mathcal{A}$ the set of reflexive, connected and acyclic binary relations on S, and $\mathcal{T}$ the set of orderings of S. A social choice function (SCF) f is a function from $\mathcal{T}^n$ to S; $f : \mathcal{T}^n \mapsto S$. A social decision rule (SDR) f is a function from $\mathcal{T}^n$ to $\mathcal{C}$; $f : \mathcal{T}^n \mapsto \mathcal{C}$. A social decision function (SDF) f is a function from $\mathcal{T}^n$ to $\mathcal{A}$; $f : \mathcal{T}^n \mapsto \mathcal{A}$.

Let $f : \mathcal{T}^n \mapsto S$ be a social choice function. Let $(\overline{R}_1, \dots, \overline{R}_n) \in \mathcal{T}^n$ be the profile of true individual orderings. $(R_1, \dots, R_n) \in \mathcal{T}^n$ is a strong equilibrium for $(\overline{R}_1, \dots, \overline{R}_n)$ iff there does not exist a coalition of individuals who by changing their strategies, while individuals outside the coalition continue to use the same strategies, can obtain an outcome which they all prefer to the outcome yielded by $(R_1, \dots, R_n)$. More formally, $(R_1, \dots, R_n) \in \mathcal{T}^n$ is a strong equilibrium for $(\overline{R}_1, \dots, \overline{R}_n)$ iff there do not exist $V \subseteq N$ and $(R_1^0, \dots, R_n^0) \in \mathcal{T}^n$ such that $[V \neq \emptyset \wedge (\forall i \in N - V)(R_i^0 = R_i) \wedge (\forall i \in V)(R_i^0 \neq R_i) \wedge (\forall i \in V)[f(R_1^0, \dots, R_n^0)\overline{P}_i f(R_1, \dots, R_n)]]$. A social choice function is strongly consistent iff for every $(\overline{R}_1, \dots, \overline{R}_n)$ there exists an $(R_1, \dots, R_n) \in \mathcal{T}^n$ which is a strong equilibrium.

In the context of social choice functions the set of social alternatives and the set of outcomes are identical, and consequently the set of strategies open to every individual is the set of orderings of the set of outcomes. This, however, is not the case with social decision rules. A social decision rule generates a reflexive and connected social binary relation for every profile of individual orderings of the set of alternatives. The choice set corresponding to the social binary relation might consist of one alternative or might consist of more than one alternative, or might even be empty. In case the choice set corresponding to the social binary relation contains more than one alternative, it will be assumed that an equal-chance random mechanism is used to select an alternative from the choice set as the final outcome, which seems appropriate, particularly in the context of social decision rules satisfying neutrality. If the choice set is empty we assume that a distinguished alternative x_0 not belonging to S (status quo or 'no decision' alternative) is selected. The results in the paper, however, are independent of this assumption and are valid even if some other procedure is adopted to break deadlock in situations where the choice set is empty. Let C^* denote the lottery corresponding to choice set C if C is nonempty, and x_0 if C is empty. Let $S^{**} = \{C^* \mid C \in 2^S\}$ be the set of outcomes. We will assume that every individual i has an ordering R_i^* over S^{**}. All logically possible R_i^* which satisfy the following two conditions would be admissible: (i) the restriction of R_i^* over S must agree with R_i, and (ii) R_i^* must be consistent with von Neumann - Morgenstern axioms of individual rationality, i.e., must be consistent with the expected utility maximization hypothesis. Let individuals' true preferences over the set of outcomes S^{**} be given by $(\overline{R}_1^*, \ldots, \overline{R}_n^*)$. Then a situation $(R_1, \ldots, R_n) \in \mathcal{T}^n$ is a strong equilibrium iff $\sim [(\exists V \subseteq N)(\exists (R_1^0, \ldots, R_n^0) \in \mathcal{T}^n)[V \neq \emptyset \wedge (\forall i \in N - V)(R_i^0 = R_i) \wedge (\forall i \in V)(R_i^0 \neq R_i) \wedge (\forall i \in V)[C(S, R^0)^* \overline{P}_i^* C(S, R)^*]]]$, i.e., iff there does not exist a coalition of individuals who by changing their strategies, while individuals outside the coalition continue to use the same strategies, can obtain an outcome which they all prefer to the outcome yielded by $(R_1, \ldots, R_n)$. A social decision rule is strongly consistent iff for every admissible $(\overline{R}_1^*, \ldots, \overline{R}_n^*)$, there exists an $(R_1, \ldots, R_n) \in \mathcal{T}^n$ which is a strong equilibrium.

The main result of the paper shows that there does not exist any neutral and monotonic non-null non-dictatorial binary social decision rule which is strongly consistent. We also show, by constructing an example of a non-dictatorial non-neutral monotonic binary social decision rule which is strongly consistent, that the result on the strong consistency of neutral and monotonic binary social decision rules cannot be generalized to cover non-neutral monotonic binary social decision rules without weakening the notion of dictatorship. The social decision rule of the example however is not interesting, as there exists an individual who, though formally not a dictator, can always obtain an outcome which is best for him by employing a suitable strategy. Thus the question whether there are any interesting non-neutral monotonic binary social decision rules which are strongly consistent remains an open one.

In the literature on the subject some connection has been noted between profiles of individual orderings satisfying value-restriction (value-restriction is sufficient for quasi-transitivity under every neutral and monotonic binary social decision rule) and the existence of strong equilibria. In the context of non-null non-dictatorial neutral and monotonic binary social decision functions, and under the assumption that individuals satisfy von Neumann - Morgenstern axioms of individual rationality, it turns out that value-restriction does not ensure the existence of a strong equilibrium. We show that for every non-dictatorial social decision function satisfying the conditions of independence of irrelevant alternatives, neutrality, monotonicity and weak Pareto-criterion there exists a profile of individual orderings satisfying value-restriction corresponding to which there is no strong equilibrium. In fact we prove a stronger result which states that for every non-dictatorial social decision function satisfying the conditions of independence of irrelevant alternatives, neutrality, monotonicity and weak Pareto-criterion there exists a profile of individual orderings satisfying strict placement restriction corresponding to which there is no strong equilibrium. Strict placement restriction implies value-restriction and is sufficient for transitivity under every neutral and monotonic binary social decision rule.

2 Notation and Definitions

The set of social alternatives and the set of individuals constituting the society are denoted by S and N respectively. We assume S and N to be finite. We denote $\#S$ and $\#N$ by s and n respectively; and assume $s \geq 3, n \geq 2$. Each individual $i \in N$ is assumed to have a binary weak preference relation R_i on S. We denote asymmetric parts of binary relations $R_i, R_i', R_i^0, R, R', R^0$, etc., by $P_i, P_i', P_i^0, P, P', P^0$, etc., respectively; and symmetric parts by $I_i, I_i', I_i^0, I, I', I^0$, etc., respectively.

We define a binary relation R on a set S to be (i) reflexive iff $(\forall x \in S)(xRx)$, (ii) connected iff $(\forall x, y \in S)(x \neq y \rightarrow xRy \vee yRx)$, (iii) anti-symmetric iff $(\forall x, y \in S)(xRy \wedge yRx \rightarrow x = y)$, (iv) acyclic iff $(\forall x_1, \ldots, x_k \in S)(x_1Px_2 \wedge \ldots \wedge x_{k-1}Px_k \rightarrow x_1Rx_k)$, where k is an integer ≥ 3, (v) quasi-transitive iff $(\forall x, y, z \in S)(xPy \wedge yPz \rightarrow xPz)$, (vi) transitive iff $(\forall x, y, z \in S)(xRy \wedge yRz \rightarrow xRz)$, (vii) an ordering iff it is reflexive, connected and transitive, and (viii) a linear ordering iff it is reflexive, connected, anti-symmetric and transitive.

Let R be a binary relation on set S. $x \in S$ is defined to be a best element of S with respect to R iff $(\forall y \in S)(xRy)$. The set of best elements in S is called its choice set and is denoted by $C(S, R)$.

We denote by $\mathcal{C}$ the set of all reflexive and connected binary relations on S, by $\mathcal{A}$ the set of all reflexive, connected and acyclic binary relations on S, by $\mathcal{T}$ the set of all reflexive, connected and transitive binary relations (orderings) on S, and by $\mathcal{L}$ the set of all reflexive, connected, antisymmetric and transitive binary relations (linear orderings) on S. A social decision rule (SDR) f is a function from $\mathcal{T}^n$ to $\mathcal{C}$; $f : \mathcal{T}^n \mapsto \mathcal{C}$. A social decision function (SDF) f is a function from $\mathcal{T}^n$ to $\mathcal{A}$; $f : \mathcal{T}^n \mapsto \mathcal{A}$. The social binary weak preference relations corresponding to $(R_1, \ldots, R_n), (R'_1, \ldots, R'_n), (R^0_1, \ldots, R^0_n)$ etc., will be denoted by R, R', R^0 etc., respectively.

An SDR satisfies (i) weak Pareto-criterion (WP) iff $(\forall (R_1, \ldots, R_n) \in \mathcal{T}^n)(\forall x, y \in S)[(\forall i \in N)(xP_iy) \rightarrow xPy]$, (ii) binariness or independence of irrelevant alternatives (I) iff $(\forall (R_1, \ldots, R_n), (R'_1, \ldots, R'_n) \in \mathcal{T}^n)(\forall x, y \in S)[(\forall i \in N)[(xR_iy \leftrightarrow xR'_iy) \wedge (yR_ix \leftrightarrow yR'_ix)] \rightarrow [(xRy \leftrightarrow xR'y) \wedge (yRx \leftrightarrow yR'x)]]$, and (iii) monotonicity (M) iff $(\forall (R_1, \ldots, R_n), (R'_1, \ldots, R'_n) \in \mathcal{T}^n)(\forall x \in S)[(\forall i \in N)[(\forall a, b \in S - \{x\})(aR_ib \leftrightarrow aR'_ib) \wedge (\forall y \in S - \{x\})[(xP_iy \rightarrow xP'_iy) \wedge (xI_iy \rightarrow xR'_iy)]] \rightarrow (\forall y \in S - \{x\})[(xPy \rightarrow xP'y) \wedge (xIy \rightarrow xR'y)]]$.

Let Φ be the set of all permutations of the set of alternatives S. Let $\phi \in \Phi$. Corresponding to a binary relation R on a set S, we define the binary relation $\phi(R)$ on S by; $(\forall x, y \in S)[\phi(x)\phi(R)\phi(y) \leftrightarrow xRy]$. An SDR satisfies neutrality (NT) iff $(\forall (R_1, \ldots, R_n), (R'_1, \ldots, R'_n) \in \mathcal{T}^n)(\forall \phi \in \Phi)[(\forall i \in N)[R'_i = \phi(R_i)] \rightarrow R' = \phi(R)]$.

It is clear from the definitions of conditions I, M and NT that an SDR $f : \mathcal{T}^n \mapsto \mathcal{C}$ satisfying condition I satisfies (i) neutrality iff $(\forall (R_1, \ldots, R_n), (R'_1, \ldots, R'_n) \in \mathcal{T}^n)(\forall x, y, z, w \in S)[(\forall i \in N)[(xR_iy \leftrightarrow zR'_iw) \wedge (yR_ix \leftrightarrow wR'_iz)] \rightarrow [(xRy \leftrightarrow zR'w) \wedge (yRx \leftrightarrow wR'z)]]$, and (ii) monotonicity iff $(\forall (R_1, \ldots, R_n), (R'_1, \ldots, R'_n) \in \mathcal{T}^n)(\forall x, y \in S)[(\forall i \in N)[(xP_iy \rightarrow xP'_iy) \wedge (xI_iy \rightarrow xR'_iy)] \rightarrow [(xPy \rightarrow xP'y) \wedge (xIy \rightarrow xR'y)]]$. An SDR is called (i) dictatorial iff $(\exists j \in N)(\forall (R_1, \ldots, R_n) \in \mathcal{T}^n)(\forall x, y \in S)(xP_jy \rightarrow xPy)$, and (ii) null iff $(\forall (R_1, \ldots, R_n) \in \mathcal{T}^n)(\forall x, y \in S)(xIy)$.

A coalition is a subset of N. A coalition V is defined to be winning iff $(\forall (R_1, \ldots, R_n) \in \mathcal{T}^n)(\forall x, y \in S)[(\forall i \in V)(xP_iy) \rightarrow xPy]$. We denote by W the set of all winning coalitions. $V \subseteq N$ is a minimal winning coalition iff V is a winning coalition and no proper subset of V is a winning coalition. The set of all minimal winning coalitions will be denoted by W_m. We define a coalition $V \subseteq N$ to be blocking iff $(\forall (R_1, \ldots, R_n) \in \mathcal{T}^n)(\forall x, y \in$

$S)[(\forall i \in V)(xP_iy) \to xRy]$. The set of all blocking coalitions will be denoted by B.

Remark 1 *Consider an SDR* $f : \mathcal{T}^n \mapsto \mathcal{C}$. *If* $V_1, V_2 \in W$ *then* $V_1 \cap V_2$ *must be non-empty, because* $V_1 \cap V_2 = \emptyset$ *would lead to a contradiction if we have for* $x, y \in S, [(\forall i \in V_1)(xP_iy) \wedge (\forall i \in V_2)(yP_ix)]$, *entailing* $(xPy \wedge yPx)$.

Remark 2 *Let* $V \in W$. *Then by the finiteness of* V *and the fact that the empty set can never be winning, it follows that there exists a nonempty* $V' \subseteq V$ *such that* $V' \in W_m$.

Let $A \subseteq S$ and let R be a binary relation on S. We define restriction of R to A, denoted by $R|A$, by $R|A = R \cap (A \times A)$. Let $\mathcal{D} \subseteq \mathcal{C}$. We define restriction of $\mathcal{D}$ to A, denoted by $\mathcal{D}|A$, by $\mathcal{D}|A = \{R|A \mid R \in \mathcal{D}\}$.

A set of three distinct alternatives will be called a triple of alternatives. Let R be an ordering of S and let $A = \{x, y, z\} \subseteq S$ be a triple of alternatives. We define R to be unconcerned over A iff $(\forall a, b \in A)(aIb)$. R is defined to be concerned over A iff it is not unconcerned over A. We define in A, according to R, x to be best iff $(xRy \wedge xRz)$, to be medium iff $(yRxRz \vee zRxRy)$, and to be worst iff $(yRx \wedge zRx)$. Furthermore, x is defined in the triple A, according to R, to be uniquely best iff $(xPy \wedge xPz)$, to be uniquely medium iff $(yPxPz \vee zPxPy)$, and to be uniquely worst iff $(yPx \wedge zPx)$.

Now we define two restrictions on sets of orderings.

Value-Restriction (VR): A set $\mathcal{D} \subset \mathcal{T}$ of orderings of S satisfies VR over a triple $A \subseteq S$ of alternatives iff $\sim (\exists$ distinct $a, b, c \in A)(\exists R^s, R^t, R^u \in \mathcal{D}|A)(R^s, R^t, R^u$ are concerned over $A \wedge aR^sbR^sc \wedge bR^tcR^ta \wedge cR^uaR^ub)$. $\mathcal{D}$ satisfies VR iff it satisfies VR over every triple of alternatives contained in S.

Strict Placement Restriction (SPR): A set $\mathcal{D} \subset \mathcal{T}$ of orderings of S satisfies SPR over a triple $A \subseteq S$ of alternatives iff $(\exists$ distinct $a, b, c \in A)[(\forall$ concerned $R \in \mathcal{D}|A)(aPb \wedge aPc) \vee (\forall$ concerned $R \in \mathcal{D}|A)(bPa \wedge cPa) \vee (\forall$ concerned $R \in \mathcal{D}|A)(bPaPc \vee cPaPb) \vee (\forall R \in \mathcal{D}|A)(aIb)]$. Less formally, a set $\mathcal{D} \subset \mathcal{T}$ of orderings of S satisfies SPR over a triple $A \subseteq S$ of alternatives iff there exists (i) an alternative such that it is uniquely best in every concerned $R \in \mathcal{D}|A$, or (ii) an alternative such that it is uniquely worst in every concerned $R \in \mathcal{D}|A$, or (iii) an alternative such that it is uniquely medium in every concerned $R \in \mathcal{D}|A$, or (iv) a pair of distinct alternatives $a, b \in A$ such that for all $R \in \mathcal{D}|A$, aIb holds. $\mathcal{D}$ satisfies SPR iff it satisfies SPR over every triple of alternatives contained in S.

Remark 3 *From the definitions of VR and SPR it is obvious that if $\mathcal{D} \subset \mathcal{T}$ satisfies SPR then $\mathcal{D}$ satisfies VR. Value-restriction is sufficient for quasi-transitivity under every neutral and monotonic binary social decision rule [Sen (1970)]; and strict placement restriction is sufficient for transitivity under every neutral and monotonic binary social decision rule [Jain (1987)].*

We assume the following choice mechanism. If the choice set contains a single element then this element emerges as the final outcome. If the choice set contains more than one element, say k, then to select one element from the choice set as the final outcome, a random mechanism is employed such that the probability of any particular element of the choice set being selected is $1/k$. If the choice set is empty we assume that a distinguished alternative $x_0 \notin S$ (status quo or 'no decision' alternative) is selected. C^* would denote the lottery corresponding to choice set C if C is nonempty, and x_0 if C is empty. Let $S^{**} = \{C^* \mid C \in 2^S\}$. Thus S^{**} is the set of all possible outcomes. We will assume that every $i \in N$ has an ordering R_i^* over S^{**}. All logically possible R_i^* which satisfy the following two conditions would be admissible: (i) the restriction of R_i^* over S must agree with R_i, and (ii) R_i^* must be consistent with von Neumann - Morgenstern axioms of individual rationality, i.e., must be consistent with expected utility maximization hypothesis. Throughout this paper we denote individual i's true preference ordering over S by $\overline{R}_i$ and over S^{**} by $\overline{R}_i^*$.

Remark 4 *The use of lexical maximin criterion for inferring individual preferences over lotteries from individual preferences over alternatives does not seem appropriate, as is shown by the following example:*

Example 1 $S = \{x_1, x_2, x_3, x_4, x_5\}, R_i = x_1 I_i x_2 I_i x_3 P_i x_4 P_i x_5$.
Consider the lotteries $L_1 = (x_1, x_2, x_3, x_4, x_5; \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ and $L_2 = (x_1, x_5; \frac{1}{2}, \frac{1}{2})$. As the probability that the outcome is at least as good as x_k with L_1 is greater than or equal to the probability that the outcome is at least as good as x_k with L_2, for all $k \in \{1, \ldots, 5\}$; and the probability that the outcome is at least as good as x_k with L_1 is greater than the probability that the outcome is at least as good as x_k with L_2, for some $k \in \{1, \ldots, 5\}$; it follows that the lottery L_1 is unconditionally better than L_2. However, lexical maximin criterion would declare L_2 to be preferable to L_1.

Let individuals' true preferences over the set S^{**} be given by $(\overline{R}_1^*, \ldots, \overline{R}_n^*)$. Then a situation $(R_1, \ldots, R_n) \in \mathcal{T}^n$ is a strong equilibrium iff $\sim [(\exists V \subseteq N)(\exists (R_1^0, \ldots, R_n^0) \in \mathcal{T}^n)[V \neq \emptyset \wedge (\forall i \in N - V)(R_i^0 = R_i) \wedge (\forall i \in V)(R_i^0 \neq R_i) \wedge (\forall i \in V)[C(S, R^0)^* \overline{P}_i^* C(S, R)^*]]]$. An SDR is strongly consistent iff for every admissible $(\overline{R}_1^*, \ldots, \overline{R}_n^*)$, there exists an

$(R_1, \ldots, R_n) \in \mathcal{T}^n$ which is a strong equilibrium.

3 Strong Consistency of Neutral and Monotonic Binary Social Decision Rules

Lemma 1 *Let social decision rule $f : \mathcal{T}^n \mapsto \mathcal{C}$ satisfy conditions I, M, NT and WP. Then f yields quasi-transitive social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{L}^n$ iff there is a unique minimal winning coalition.*

Proof: *Necessity*
Suppose there exist two distinct minimal winning coalitions V_1 and V_2. Let $x, y, z \in S$ and consider any $(R_1, \ldots, R_n) \in \mathcal{L}^n$ with the following restriction over $\{x, y, z\}$:
$(\forall i \in V_1 \cap V_2)(xP_iyP_iz)$
$(\forall i \in V_1 - V_2)(yP_izP_ix)$
$(\forall i \in N - V_1)(zP_ixP_iy)$.
We must have xPy and yPz as $[(\forall i \in V_2)(xP_iy) \wedge (\forall i \in V_1)(yP_iz)]$. xPz would imply, by conditions I, M, and NT, that $V_1 \cap V_2$ is a winning coalition, which would contradict the hypothesis that V_1 and V_2 are distinct minimal winning coalitions. So we must have $\sim (xPz)$. This establishes that if an SDR satisfying conditions I, M, NT and WP yields quasi-transitive social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{L}^n$ then there is at most one minimal winning coalition. However, by condition WP and Remark 2, there is at least one minimal winning coalition. Therefore, it follows that if an SDR satisfying conditions I, M, NT and WP yields quasi-transitive social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{L}^n$ then there is a unique minimal winning coalition.
Sufficiency
Let there be a unique minimal winning coalition V. Consider any $x, y, z \in S$ and any $(R_1, \ldots, R_n) \in \mathcal{L}^n$ such that xPy and yPz. Designate by V_1 and V_2 the sets $\{i \in N : xP_iy\}$ and $\{i \in N : yP_iz\}$ respectively. V_1 and V_2 are winning coalitions as a consequence of conditions I, M, and NT. Therefore there exist $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$ such that V_1' and V_2' are minimal winning coalitions. As there is a unique minimal winning coalition we conclude that $V_1' = V_2' = V$. So $(\forall i \in V)(xP_iz)$, which implies xPz. This establishes the lemma.

Remark 5 *The above lemma can easily be generalized as follows:*

Let social decision rule $f : \mathcal{T}^n \mapsto \mathcal{C}$ satisfy conditions I, M and WP. Then f yields quasi-transitive social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{L}^n$ iff there is a unique minimal winning coalition.

The generalized version of the lemma is analogous to Gibbard Theorem [Gibbard (1969)], which states that if an SDR $f : \mathcal{T}^n \mapsto \mathcal{C}$ satisfying conditions I and WP yields quasi-transitive social binary weak preference relation for every $(R_1, \dots, R_n) \in \mathcal{T}^n$ then there is a unique minimal winning coalition. The two results, however, are logically independent of each other. Furthermore, for social decision rules satisfying conditions I and WP, in general, it is not true that the existence of a unique minimal winning coalition implies that the social binary weak preference relation is quasi-transitive for every $(R_1, \dots, R_n) \in \mathcal{T}^n$.

Lemma 2 *Let social decision rule $f : \mathcal{T}^n \mapsto \mathcal{C}$ satisfy conditions I, M, NT and WP. Then, a necessary condition for the strong consistency of f is that it yield quasi-transitive social binary weak preference relation for every $(R_1, \dots, R_n) \in \mathcal{L}^n$.*

Proof: Suppose f does not yield quasi-transitive social binary weak preference relation for every $(R_1, \dots, R_n) \in \mathcal{L}^n$. Then by Lemma 1, $(\exists V_1, V_2)[V_1, V_2 \in W_m \wedge V_1 \neq V_2]$. By conditions I, M, and NT it follows that $N - (V_1 \cap V_2)$ is a blocking coalition. Let $S = \{x, y, z, w_1, \dots, w_{s-3}\}$. Consider the following profile $(\overline{R}_1, \dots, \overline{R}_n)$ of true individual orderings:

$(\forall i \in V_1 \cap V_2)(x\overline{P}_i y\overline{P}_i z\overline{P}_i w_1\overline{P}_i \dots \overline{P}_i w_{s-3})$
$(\forall i \in V_1 - V_2)(z\overline{P}_i x\overline{P}_i y\overline{P}_i w_1\overline{P}_i \dots \overline{P}_i w_{s-3})$
$(\forall i \in N - V_1)(y\overline{P}_i z\overline{P}_i x\overline{P}_i w_1\overline{P}_i \dots \overline{P}_i w_{s-3})$.
Furthermore assume that:
$[(\forall i \in N)(\forall a \in S)(a\overline{P}_i^* x_0) \wedge (\forall i \in V_1 \cap V_2)(y\overline{P}_i^* \{x, y, z\}^* \overline{P}_i^* \{x, z\}^*)]$.

Now we show that no $(R_1, \dots, R_n) \in \mathcal{T}^n$ can be a strong equilibrium for the above situation.

(i) Consider any $(R_1, \dots, R_n) \in \mathcal{T}^n$ which yields x as the outcome. Construct $(R_1^0, \dots, R_n^0) \in \mathcal{T}^n$ as follows:

$(\forall i \in V_1 \cap V_2)(R_i^0 = R_i)$
$(\forall i \in N - (V_1 \cap V_2))(\forall a \in S - \{z\})(zP_i^0 a)$
$(\forall i \in N - (V_1 \cap V_2))(R_i^0|S - \{z\} = R_i|S - \{z\})$.

As $N - (V_1 \cap V_2)$ is a blocking coalition, $(\forall i \in N - (V_1 \cap V_2))(\forall a \in S - \{z\})(zP_i^0 a)$ implies that $(\forall a \in S - \{z\})(zR^0 a)$. Therefore $z \in C(S, R^0)$. Furthermore no $a \in S - \{x, z\}$ can belong to $C(S, R^0)$, as a consequence of conditions I and M. Therefore the outcome associated with $(R_1^0, \dots, R_n^0)$ is z or $\{x, z\}^*$. As $(\forall i \in N - (V_1 \cap V_2))[z\overline{P}_i x \wedge \{x, z\}^* \overline{P}_i^* x]$, we conclude that $(R_1, \dots, R_n)$ is not a strong equilibrium for the situation under consideration.

(ii) Consider any $(R_1, \ldots, R_n) \in \mathcal{T}^n$ which yields z or $\{x, z\}^*$ or $\{y, z\}^*$ or $\{x, y, z\}^*$ as the outcome. Construct $(R_1^0, \ldots, R_n^0) \in \mathcal{T}^n$ as follows:

$(\forall i \in N - V_2)(R_i^0 = R_i)$
$(\forall i \in V_2)(\forall a \in S - \{y\})(yP_i^0 a)$
$(\forall i \in V_2)(R_i^0 | S - \{y\} = R_i | S - \{y\})$.

As V_2 is a winning coalition, $(R_1^0, \ldots, R_n^0)$ yields y as the outcome. As $(\forall i \in V_2)[y\overline{P}_i z \wedge y\overline{P}_i^* \{x, z\}^* \wedge y\overline{P}_i^* \{y, z\}^* \wedge y\overline{P}_i^* \{x, y, z\}^*]$, we conclude that $(R_1, \ldots, R_n)$ is not a strong equilibrium for the situation under consideration.

(iii) Consider any $(R_1, \ldots, R_n) \in \mathcal{T}^n$ which yields y or $\{x, y\}^*$ as the outcome. Construct $(R_1^0, \ldots, R_n^0) \in \mathcal{T}^n$ as follows :

$(\forall i \in N - V_1)(R_i^0 = R_i)$
$(\forall i \in V_1)(\forall a \in S - \{x\})(xP_i^0 a)$
$(\forall i \in V_1)(R_i^0 | S - \{x\} = R_i | S - \{x\})$.

As V_1 is a winning coalition, $(R_1^0, \ldots, R_n^0)$ yields x as the outcome. As $(\forall i \in V_1)[x\overline{P}_i y \wedge x\overline{P}_i^* \{x, y\}^*]$, we conclude that $(R_1, \ldots, R_n)$ is not a strong equilibrium for the situation under consideration.

(iv) Consider any $(R_1, \ldots, R_n) \in \mathcal{T}^n$ which yields B^* as the outcome, where $B \subseteq \{w_1, \ldots, w_{s-3}\}$ and $B \neq \emptyset$. Construct $(R_1^0, \ldots, R_n^0) \in \mathcal{T}^n$ as follows:

$(\forall i \in N)(\forall a \in S - \{x\})(xP_i^0 a)$
$(\forall i \in N)(R_i^0 | S - \{x\} = R_i | S - \{x\})$.

As N is a winning coalition, $(R_1^0, \ldots, R_n^0)$ yields x as the outcome. As $(\forall i \in N)[x\overline{P}_i^* B^*]$, we conclude that $(R_1, \ldots, R_n)$ is not a strong equilibrium for the situation under consideration.

(v) Consider any $(R_1, \ldots, R_n) \in \mathcal{T}^n$ which yields $(A \cup B)^*$ as the outcome, where $A \subseteq \{x, y, z\}, B \subseteq \{w_1, \ldots, w_{s-3}\}, A \neq \emptyset$ and $B \neq \emptyset$. Construct $(R_1^0, \ldots, R_n^0) \in \mathcal{T}^n$ as follows:

$(\forall i \in N)(\forall a, b \in A)(aI_i^0 b)$
$(\forall i \in N)(\forall a \in A)(\forall b \in S - A)(aP_i^0 b)$
$(\forall i \in N)(R_i^0|S - A = R_i|S - A)$.

By conditions WP and NT we conclude that outcome associated with $(R_1^0, \ldots, R_n^0)$ is A^*. As $(\forall i \in N)[A^* \overline{P}_i^* (A \cup B)^*]$, we conclude that $(R_1, \ldots, R_n)$ is not a strong equilibrium for the situation under consideration.

(vi) Consider any $(R_1, \ldots, R_n) \in \mathcal{T}^n$ which yields x_0 as the outcome. Construct $(R_1^0, \ldots, R_n^0) \in \mathcal{T}^n$ as follows:

$(\forall i \in N)(\forall a \in S - \{x\})(xP_i^0 a)$
$(\forall i \in N)(R_i^0|S - \{x\} = R_i|S - \{x\})$.

As N is a winning coalition, $(R_1^0, \ldots, R_n^0)$ yields x as the outcome. As $(\forall i \in N)[x\overline{P}_i^* x_0]$, we conclude that $(R_1, \ldots, R_n)$ is not a strong equilibrium for the situation under consideration.

(i)-(vi) establish that there is no $(R_1, \ldots, R_n) \in \mathcal{T}^n$ which is a strong equilibrium for the situation under consideration, which establishes the lemma.

Remark 6 *If it is assumed that when the choice set is empty the outcome is some $A^*, A \subseteq S, A \neq \emptyset$, not necessarily the same A^* for every situation involving the empty choice set, then (i)-(v) establish the lemma.*

Lemma 3 *Let social decision rule $f : \mathcal{T}^n \mapsto \mathcal{C}$ satisfy conditions I, M, NT and WP. Then f yields transitive social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{L}^n$ iff there is a unique minimal winning coalition consisting of a single individual.*

Proof: *Necessity*
Let f yield transitive social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{L}^n$. By Lemma 1 this implies that there is a unique minimal winning coalition V. Suppose V contains more than one individual. Then there exist nonempty disjoint V_1 and V_2 such that $V_1 \cup V_2 = V$. Let $x, y, z \in S$ and consider any $(R_1, \ldots, R_n) \in \mathcal{L}^n$ with the following restriction over $\{x, y, z\}$:
$(\forall i \in V_1)(xP_i yP_i z)$
$(\forall i \in V_2)(zP_i xP_i y)$
$(\forall i \in N - V)(yP_i zP_i x)$.
We obtain xPy as $[(\forall i \in V)(xP_i y)]$. As zPy would imply, by conditions I, M, and NT, that V_2 is a winning coalition, contradicting the fact that V is the unique minimal winning

coalition, we conclude that yRz holds. xPy and yRz imply xPz, as social binary weak preference relation is transitive for every $(R_1, \ldots, R_n) \in \mathcal{L}^n$. xPz in turn implies, by conditions I, M, and NT, that V_1 is a winning coalition, contradicting the fact that V is the unique minimal winning coalition. This contradiction establishes that V consists of a single individual.

Sufficiency

Suppose there is a unique minimal winning coalition consisting of a single individual, say, j. Consider any $x, y, z \in S$ and any $(R_1, \ldots, R_n) \in \mathcal{L}^n$ such that xRy and yRz. yP_jx would imply yPx, so we must have xP_jy. By a similar argument it follows that yP_jz holds. xP_jy and yP_jz imply xP_jz, which in turn implies xPz. This establishes that xRz holds, which concludes the proof.

Remark 7 *The above lemma can easily be generalized as follows:*
Let social decision rule $f : \mathcal{T}^n \mapsto \mathcal{C}$ satisfy conditions I, M and WP. Then f yields transitive social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{L}^n$ iff there is a unique minimal winning coalition consisting of a single individual.

The generalized version of lemma 3 is analogous to Arrow Impossibility Theorem which states that if an SDR $f : \mathcal{T}^n \mapsto \mathcal{C}$ satisfying conditions I and WP yields transitive social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{T}^n$ then there is a unique minimal winning coalition consisting of a single individual, i. e., there is a dictator. The two results, however, are logically independent of each other. Furthermore, for social decision rules satisfying conditions I and WP, in general, it is not true that the existence of a unique minimal winning coalition consisting of a single individual implies that the social binary weak preference relation is transitive for every $(R_1, \ldots, R_n) \in \mathcal{T}^n$.

Lemma 4 *Let social decision rule $f : \mathcal{T}^n \mapsto \mathcal{C}$ satisfy conditions I, M, NT and WP, and yield quasi-transitive social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{L}^n$. Then, a necessary condition for the strong consistency of f is that it yield transitive social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{L}^n$.*

Proof: Suppose f does not yield transitive social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{L}^n$. Then by Lemmas 1 and 3 it follows that there is a unique minimal winning coalition V consisting of more than one individual. Let $V = V_1 \cup V_2$, such that V_1 and V_2 are nonempty and disjoint. As V is the unique minimal winning coalition, by conditions I, M, and NT, it follows that V_1 and $N - V_1$ are blocking coalitions. Let $S = \{x, y, z, w_1, \ldots, w_{s-3}\}$. Consider the following profile $(\overline{R}_1, \ldots, \overline{R}_n)$ of true individual orderings:

$(\forall i \in V_1)(x\overline{P}_i y\overline{P}_i z\overline{P}_i w_1\overline{P}_i \ldots \overline{P}_i w_{s-3})$
$(\forall i \in N - V_1) : (z\overline{P}_i y\overline{P}_i x\overline{P}_i w_1\overline{P}_i \ldots \overline{P}_i w_{s-3})$.
Furthermore assume that $(\forall i \in N)[(\forall a \in S)(a\overline{P}_i^* x_0) \wedge (y\overline{P}_i^*\{x, y, z\}^*\overline{P}_i^*\{x, z\}^*)]$.

Now we show that no $(R_1, \ldots, R_n) \in \mathcal{T}^n$ can be a strong equilibrium for the above situation.

(i) Consider any $(R_1, \ldots, R_n) \in \mathcal{T}^n$ which yields y or z or $\{y, z\}^*$ as the outcome. Construct $(R_1^0, \ldots, R_n^0) \in \mathcal{T}^n$ as follows :

$(\forall i \in N - V_1)(R_i^0 = R_i)$
$(\forall i \in V_1)(\forall a \in S - \{x\})(xP_i^0 a)$
$(\forall i \in V_1)(R_i^0|S - \{x\} = R_i|S - \{x\})$.

As V_1 is a blocking coalition and $(\forall i \in V_1)(\forall a \in S - \{x\})(xP_i^0 a)$, it follows that $x \in C(S, R^0)$. Furthermore no $a \in S - [C(S, R) \cup \{x\}]$ can belong to $C(S, R^0)$, as a consequence of conditions I and M. Therefore the outcome associated with $(R_1^0, \ldots, R_n^0)$ is x or $\{x, y\}^*$ in case $(R_1, \ldots, R_n)$ yields y; is x or $\{x, z\}^*$ in case $(R_1, \ldots, R_n)$ yields z; and is x or $\{x, y\}^*$ or $\{x, z\}^*$ or $\{x, y, z\}^*$ in case $(R_1, \ldots, R_n)$ yields $\{y, z\}^*$. As $(\forall i \in V_1)[[x\overline{P}_i y \wedge \{x, y\}^*\overline{P}_i^* y] \wedge [x\overline{P}_i z \wedge \{x, z\}^*\overline{P}_i^* z] \wedge [x\overline{P}_i^*\{y, z\}^* \wedge \{x, y\}^*\overline{P}_i^*\{y, z\}^* \wedge \{x, z\}^*\overline{P}_i^*\{y, z\}^* \wedge \{x, y, z\}^*\overline{P}_i^*\{y, z\}^*]]$, we conclude that $(R_1, \ldots, R_n)$ is not a strong equilibrium for the situation under consideration.

(ii) Consider any $(R_1, \ldots, R_n) \in \mathcal{T}^n$ which yields x or $\{x, y\}^*$ as the outcome. Construct $(R_1^0, \ldots, R_n^0) \in \mathcal{T}^n$ as follows :

$(\forall i \in V_1)(R_i^0 = R_i)$
$(\forall i \in N - V_1)(\forall a \in S - \{z\})(zP_i^0 a)$
$(\forall i \in N - V_1)(R_i^0|S - \{z\} = R_i|S - \{z\})$.

As $N - V_1$ is a blocking coalition, and $(\forall i \in N - V_1)(\forall a \in S - \{z\})(zP_i^0 a)$, it follows that $z \in C(S, R^0)$. Furthermore no $a \in S - [C(S, R) \cup \{z\}]$ can belong to $C(S, R^0)$, as a consequence of conditions I and M. Therefore the outcome associated with $(R_1^0, \ldots, R_n^0)$ is z or $\{x, z\}^*$ in case $(R_1, \ldots, R_n)$ yields x; and is z or $\{x, z\}^*$ or $\{y, z\}^*$ or $\{x, y, z\}^*$ in case $(R_1, \ldots, R_n)$ yields $\{x, y\}^*$. As $(\forall i \in N - V_1)[[z\overline{P}_i x \wedge \{x, z\}^*\overline{P}_i^* x] \wedge [z\overline{P}_i^*\{x, y\}^* \wedge \{x, z\}^*\overline{P}_i^*\{x, y\}^* \wedge \{y, z\}^*\overline{P}_i^*\{x, y\}^* \wedge \{x, y, z\}^*\overline{P}_i^*\{x, y\}^*]]$, we conclude that $(R_1, \ldots, R_n)$ is not a strong equilibrium for the situation under consideration.

(iii) Consider any $(R_1, \dots, R_n) \in \mathcal{T}^n$ which yields $\{x, z\}^*$ or $\{x, y, z\}^*$ or B^* as the outcome, where $B \subseteq \{w_1, \dots, w_{s-3}\}$ and $B \neq \emptyset$. Construct $(R_1^0, \dots, R_n^0) \in \mathcal{T}^n$ as follows:

$(\forall i \in N)(\forall a \in S - \{y\})(yP_i^0 a)$
$(\forall i \in N)(R_i^0 | S - \{y\} = R_i | S - \{y\})$.

As N is a winning coalition, $(R_1^0, \dots, R_n^0)$ yields y as the outcome. As $(\forall i \in N)[y\overline{P}_i^* \{x, z\}^* \wedge y\overline{P}_i^* \{x, y, z\}^* \wedge y\overline{P}_i^* B^*]$, we conclude that $(R_1, \dots, R_n)$ is not a strong equilibrium for the situation under consideration.

(iv) Consider any $(R_1, \dots, R_n) \in \mathcal{T}^n$ which yields $(A \cup B)^*$ as the outcome, where $A \subseteq \{x, y, z\}, B \subseteq \{w_1, \dots, w_{s-3}\}, A \neq \emptyset$ and $B \neq \emptyset$. Construct $(R_1^0, \dots, R_n^0) \in \mathcal{T}^n$ as follows:

$(\forall i \in N)(\forall a, b \in A)(aI_i^0 b)$
$(\forall i \in N)(\forall a \in A)(\forall b \in S - A)(aP_i^0 b)$
$(\forall i \in N)(R_i^0 | S - A = R_i | S - A)$.

By conditions WP and NT we conclude that outcome associated with $(R_1^0, \dots, R_n^0)$ is A^*. As $(\forall i \in N)[A^* \overline{P}_i^* (A \cup B)^*]$, we conclude that $(R_1, \dots, R_n)$ is not a strong equilibrium for the situation under consideration.

(v) Consider any $(R_1, \dots, R_n) \in \mathcal{T}^n$ which yields x_0 as the outcome. Construct $(R_1^0, \dots, R_n^0) \in \mathcal{T}^n$ as follows:

$(\forall i \in N)(\forall a \in S - \{x\})(xP_i^0 a)$
$(\forall i \in N)(R_i^0 | S - \{x\} = R_i | S - \{x\})$.

As N is a winning coalition, $(R_1^0, \dots, R_n^0)$ yields x as the outcome. As $(\forall i \in N)[x\overline{P}_i^* x_0]$, we conclude that $(R_1, \dots, R_n)$ is not a strong equilibrium for the situation under consideration.

(i)-(v) establish the lemma.

Remark 8 *If it is assumed that when the choice set is empty the outcome is some $A^*, A \subseteq S, A \neq \emptyset$, not necessarily the same A^* for every situation involving the empty choice set, then (i)-(iv) establish the lemma.*

Theorem 1 *There does not exist any neutral and monotonic non-null non-dictatorial binary social decision rule $f : \mathcal{T}^n \mapsto \mathcal{C}$ which is strongly consistent.*

Proof: If an SDR, satisfying conditions I, M and NT, violates WP then it must be null. By Lemmas 2 and 4, an SDR $f : \mathcal{T}^n \mapsto \mathcal{C}$ satisfying conditions I, M, NT and WP is strongly consistent only if it yields transitive social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{L}^n$. However, by Lemma 3, every SDR $f : \mathcal{T}^n \mapsto C$ satisfying conditions I, M, NT and WP, which yields transitive social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{L}^n$ is dictatorial. This establishes the theorem.

The notion of strong equilibrium used in this paper is: Let individuals' preferences over the set of outcomes S^{**} be given by $(\overline{R}_1^*, \ldots, \overline{R}_n^*)$. Then $(R_1, \ldots, R_n) \in \mathcal{T}^n$ is a strong equilibrium for $(\overline{R}_1^*, \ldots, \overline{R}_n^*)$ iff $\sim [(\exists V \subseteq N)(\exists (R_1^0, \ldots, R_n^0) \in \mathcal{T}^n)[V \neq \emptyset \wedge (\forall i \in N - V)(R_i^0 = R_i) \wedge (\forall i \in V)(R_i^0 \neq R_i) \wedge (\forall i \in V)[C(S, R^0)^* \overline{P}_i^* C(S, R)^*]]]$.

A stricter notion of strong equilibrium can be defined as follows :

Let individuals' preferences over the set of outcomes S^{**} be given by $(\overline{R}_1^*, \ldots, \overline{R}_n^*)$. Then $(R_1, \ldots, R_n) \in \mathcal{T}^n$ is a strong equilibrium in the strict sense for $(\overline{R}_1^*, \ldots, \overline{R}_n^*)$ iff $\sim [(\exists V \subseteq N)(\exists (R_1^0, \ldots, R_n^0) \in \mathcal{T}^n)[V \neq \emptyset \wedge (\forall i \in N - V)(R_i^0 = R_i) \wedge (\forall i \in V)(R_i^0 \neq R_i) \wedge (\forall i \in V)[C(S, R^0)^* \overline{R}_i^* C(S, R)^*] \wedge (\exists i \in V)[C(S, R^0)^* \overline{P}_i^* C(S, R)^*]]]$.

Remark 9 *It should be noted that Theorem 1 cannot be generalized to cover non-neutral SDRs without weakening the notion of dictatorship, as the following example shows.*

Example 2 $S = \{x, y, z\}, N = \{1, 2\}$.
SDR is characterized as follows:
(a) $[xPy \leftrightarrow xR_1y] \wedge [yPx \leftrightarrow yP_1x]$
$(b)[yPz \leftrightarrow yR_1z] \wedge [zPy \leftrightarrow zP_1y]$
$(c)[xPz \leftrightarrow xP_1z \wedge xP_2z] \wedge [zPx \leftrightarrow zP_1x \wedge zP_2x] \wedge [xIz \leftrightarrow \sim (xPz) \wedge \sim (zPx)]$.
Now we show that the above SDR is strongly consistent (even if one uses the stricter notion of strong equilibrium), i.e., for every configuration of true individual preferences over the set of outcomes there is a strong equilibrium.
(i) If $[[C(S, \overline{R}_1) = x] \vee [C(S, \overline{R}_1) = \{x, y\} \wedge x\overline{R}_2 y] \vee [C(S, \overline{R}_1) = \{x, z\} \wedge x\overline{R}_2 z] \vee [C(S, \overline{R}_1) = \{x, y, z\} \wedge x \in C(S, \overline{R}_2)]]$ *then* $[R_1 = xP_1yP_1z \wedge R_2 = \overline{R}_2]$ *is a strong equilibrium.*
(ii) If $[[C(S, \overline{R}_1) = y] \vee [C(S, \overline{R}_1) = \{x, y\} \wedge y\overline{P}_2 x] \vee [C(S, \overline{R}_1) = \{y, z\} \wedge y\overline{R}_2 z] \vee [C(S, \overline{R}_1) = \{x, y, z\} \wedge y \in C(S, \overline{R}_2)]]$ *then* $[R_1 = yP_1zP_1x \wedge R_2 = \overline{R}_2]$ *is a strong equilibrium.*
(iii) If $[[C(S, \overline{R}_1) = z] \vee [C(S, \overline{R}_1) = \{x, z\} \wedge z\overline{P}_2 x] \vee [C(S, \overline{R}_1) = \{y, z\} \wedge z\overline{P}_2 y] \vee [C(S, \overline{R}_1) = \{x, y, z\} \wedge z \in C(S, \overline{R}_2)]]$ *then* $[R_1 = zP_1yP_1x \wedge R_2 = \overline{R}_2]$ *is a strong equilibrium.*

4 Restricted Preferences and Existence of Strong Equilibria

Theorem 2 *For every non-dictatorial social decision function $f : \mathcal{T}^n \mapsto \mathcal{A}$ satisfying the conditions of independence of irrelevant alternatives, neutrality, monotonicity and weak Pareto-criterion there exists an $(\overline{R}_1^*, \ldots, \overline{R}_n^*)$ such that (i) $(\overline{R}_1^*|S, \ldots, \overline{R}_n^*|S) = (\overline{R}_1, \ldots, \overline{R}_n) \in \mathcal{L}^n$, (ii) $(\overline{R}_1, \ldots, \overline{R}_n)$ satisfies strict placement restriction, and (iii) there is no strong equilibrium corresponding to $(\overline{R}_1^*, \ldots, \overline{R}_n^*)$.*

Proof: By condition WP and Remark 2, W_m is nonempty. There are two cases to be considered depending on whether W_m is a singleton or not.

Case (a)

Suppose there exist distinct minimal winning coalitions V_1 and V_2. $V_1 \cap V_2$ is nonempty by Remark 1. $N - (V_1 \cap V_2)$ is nonempty as V_1 and V_2 are distinct minimal winning coalitions. First we note that $N - (V_1 \cap V_2)$ cannot be a winning coalition, otherwise there would exist winning coalitions V_1, V_2, and $N - (V_1 \cap V_2)$ with empty intersection. However, it is impossible for any three winning coalitions to have empty intersection as $\#S \geq 3$ and f yields acyclic social binary weak preference relation for every $(R_1, \ldots, R_n) \in \mathcal{T}^n$. $N - (V_1 \cap V_2)$ not being a winning coalition implies, by conditions I, M and NT, that $(V_1 \cap V_2)$ is a blocking coalition. Furthermore, $(V_1 \cap V_2)$ cannot be a winning coalition as V_1 and V_2 are distinct minimal winning coalitions. Consequently, by conditions I, M and NT, $N - (V_1 \cap V_2)$ is a blocking coalition. Thus both $V_1 \cap V_2$ and $N - (V_1 \cap V_2)$ are blocking coalitions and neither is a winning coalition.

Let $S = \{x, y, z, w_1, \ldots, w_{s-3}\}$. Consider the following profile $(\overline{R}_1, \ldots, \overline{R}_n)$ of true individual orderings satisfying SPR:

$(\forall i \in (V_1 \cap V_2))(x\overline{P}_i y\overline{P}_i z\overline{P}_i w_1\overline{P}_i \ldots \overline{P}_i w_{s-3})$
$(\forall i \in N - (V_1 \cap V_2))(z\overline{P}_i y\overline{P}_i x\overline{P}_i w_1\overline{P}_i \ldots \overline{P}_i w_{s-3})$.
Furthermore assume that:
$(\forall i \in N)[y\overline{P}_i^*\{x, y, z\}^*\overline{P}_i^*\{x, z\}^*]$.

Now we show that no $(R_1, \ldots, R_n) \in \mathcal{T}^n$ can be a strong equilibrium for the above situation.

(i) Consider any $(R_1, \ldots, R_n) \in \mathcal{T}^n$ which yields y or z or $\{y, z\}^*$ as the outcome. Construct $(R_1^0, \ldots, R_n^0) \in \mathcal{T}^n$ as follows :

$(\forall i \in N - (V_1 \cap V_2))(R_i^0 = R_i)$
$(\forall i \in V_1 \cap V_2)(\forall a \in S - x)(xP_i^0 a)$
$(\forall i \in V_1 \cap V_2)(R_i^0 | S - \{x\} = R_i | S - \{x\})$.

As $(V_1 \cap V_2)$ is a blocking coalition and $(\forall i \in V_1 \cap V_2)(\forall a \in S - \{x\})(xP_i^0 a)$, it follows that $x \in C(S, R^0)$. Furthermore no $a \in S - [C(S, R) \cup \{x\}]$ can belong to $C(S, R^0)$, as a consequence of conditions I and M. Therefore the outcome associated with $(R_1^0, \ldots, R_n^0)$ is x or $\{x, y\}^*$ in case $(R_1, \ldots, R_n)$ yields y; is x or $\{x, z\}^*$ in case $(R_1, \ldots, R_n)$ yields z; and is x or $\{x, y\}^*$ or $\{x, z\}^*$ or $\{x, y, z\}^*$ in case $(R_1, \ldots, R_n)$ yields $\{y, z\}^*$. As $(\forall i \in V_1 \cap V_2)[[x\overline{P}_i y \wedge \{x, y\}^* \overline{P}_i^* y] \wedge [x\overline{P}_i z \wedge \{x, z\}^* \overline{P}_i^* z] \wedge [x\overline{P}_i^* \{y, z\}^* \wedge \{x, y\}^* \overline{P}_i^* \{y, z\}^* \wedge \{x, z\}^* \overline{P}_i^* \{y, z\}^* \wedge \{x, y, z\}^* \overline{P}_i^* \{y, z\}^*]]$, we conclude that $(R_1, \ldots, R_n)$ is not a strong equilibrium for the situation under consideration.

(ii) Consider any $(R_1, \ldots, R_n) \in \mathcal{T}^n$ which yields x or $\{x, y\}^*$ as the outcome. Construct $(R_1^0, \ldots, R_n^0) \in \mathcal{T}^n$ as follows:

$(\forall i \in V_1 \cap V_2)(R_i^0 = R_i)$
$(\forall i \in N - (V_1 \cap V_2))(\forall a \in S - \{z\})(zP_i^0 a)$
$(\forall i \in N - (V_1 \cap V_2))(R_i^0 | S - \{z\} = R_i | S - \{z\})$.

As $N - (V_1 \cap V_2)$ is a blocking coalition, and $(\forall i \in N - (V_1 \cap V_2))(\forall a \in S - \{z\})(zP_i^0 a)$, it follows that $z \in C(S, R^0)$. Furthermore no $a \in S - [C(S, R) \cup \{z\}]$ can belong to $C(S, R^0)$, as a consequence of conditions I and M. Therefore the outcome associated with $(R_1^0, \ldots, R_n^0)$ is z or $\{x, z\}^*$ in case $(R_1, \ldots, R_n)$ yields x; and is z or $\{x, z\}^*$ or $\{y, z\}^*$ or $\{x, y, z\}^*$ in case $(R_1, \ldots, R_n)$ yields $\{x, y\}^*$. As $(\forall i \in N - (V_1 \cap V_2))[[z\overline{P}_i x \wedge \{x, z\}^* \overline{P}_i^* x] \wedge [z\overline{P}_i^* \{x, y\}^* \wedge \{x, z\}^* \overline{P}_i^* \{x, y\}^* \wedge \{y, z\}^* \overline{P}_i^* \{x, y\}^* \wedge \{x, y, z\}^* \overline{P}_i^* \{x, y\}^*]]$, we conclude that $(R_1, \ldots, R_n)$ is not a strong equilibrium for the situation under consideration.

(iii) Consider any $(R_1, \ldots, R_n) \in \mathcal{T}^n$ which yields $\{x, z\}^*$ or $\{x, y, z\}^*$ or B^* as the outcome, where $B \subseteq \{w_1, \ldots, w_{s-3}\}$ and $B \neq \emptyset$. Construct $(R_1^0, \ldots, R_n^0) \in \mathcal{T}^n$ as follows:

$(\forall i \in N)(\forall a \in S - \{y\})(yP_i^0 a)$
$(\forall i \in N)(R_i^0 | S - \{y\} = R_i | S - \{y\})$.

As N is a winning coalition, $(R_1^0, \ldots, R_n^0)$ yields y as the outcome. As $(\forall i \in N)[y\overline{P}_i^* \{x, z\}^* \wedge$

$y\overline{P}_i^*\{x,y,z\}^* \wedge y\overline{P}_i^* B^*]$, we conclude that $(R_1,\ldots,R_n)$ is not a strong equilibrium for the situation under consideration.

(iv) Consider any $(R_1,\ldots,R_n) \in \mathcal{T}^n$ which yields $(A \cup B)^*$ as the outcome, where $A \subseteq \{x,y,z\}, B \subseteq \{w_1,\ldots,w_{s-3}\}, A \neq \emptyset$ and $B \neq \emptyset$. Construct $(R_1^0,\ldots,R_n^0) \in \mathcal{T}^n$ as follows:

$(\forall i \in N)(\forall a,b \in A)(aI_i^0 b)$
$(\forall i \in N)(\forall a \in A)(\forall b \in S-A)(aP_i^0 b)$
$(\forall i \in N)(R_i^0|S-A = R_i|S-A)$.

By conditions WP and NT we conclude that outcome associated with $(R_1^0,\ldots,R_n^0)$ is A^*. As $(\forall i \in N)[A^*\overline{P}_i^*(A \cup B)^*]$, we conclude that $(R_1,\ldots,R_n)$ is not a strong equilibrium for the situation under consideration.

(i)-(iv) establish the claim that there does not exist any $(R_1,\ldots,R_n) \in \mathcal{T}^n$ which is a strong equilibrium for the situation under consideration.

Case (b)

Suppose there is a unique minimal winning coalition V. V must contain more than one individual as f is non-dictatorial. Let $V = V_1 \cup V_2$, such that V_1 and V_2 are nonempty and disjoint. As V is the unique minimal winning coalition, by conditions I, M, and NT, it follows that V_1 and $N - V_1$ are blocking coalitions. Let $S = \{x,y,z,w_1,\ldots,w_{s-3}\}$. Consider the following profile $(\overline{R}_1,\ldots,\overline{R}_n)$ of true individual orderings satisfying SPR:

$(\forall i \in V_1)(x\overline{P}_i y\overline{P}_i z\overline{P}_i w_1\overline{P}_i \ldots \overline{P}_i w_{s-3})$
$(\forall i \in N-V_1)(z\overline{P}_i y\overline{P}_i x\overline{P}_i w_1\overline{P}_i \ldots \overline{P}_i w_{s-3})$.
Furthermore assume that:
$(\forall i \in N)[y\overline{P}_i^*\{x,y,z\}^*\overline{P}_i^*\{x,z\}^*]$.

By replacing $V_1 \cap V_2$ by V_1 in (i)-(iv) of case (a) one obtains the proof that there does not exist any $(R_1,\ldots,R_n) \in \mathcal{T}^n$ which is a strong equilibrium for the above situation. This establishes the theorem.

Corollary 1 *For every non-dictatorial social decision function $f : \mathcal{T}^n \mapsto \mathcal{A}$ satisfying the conditions of independence of irrelevant alternatives, neutrality, monotonicity and weak Pareto-criterion there exists an $(\overline{R}_1^*,\ldots,\overline{R}_n^*)$ such that (i) $(\overline{R}_1^*|S,\ldots,\overline{R}_n^*|S) = (\overline{R}_1,\ldots,\overline{R}_n) \in \mathcal{L}^n$, (ii) $(\overline{R}_1,\ldots,\overline{R}_n)$ satisfies value restriction, and (iii) there is no strong* *equilibrium corresponding to* $(\overline{R}_1^*, \ldots, \overline{R}_n^*)$.

Proof: Follows immediately from Theorem 2 in view of the fact that SPR implies VR.

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