Abstract: Chapter 1

The Chapter introduces the subject matter of the monograph, namely the derivation of Inada-type conditions for transitivity and quasi-transitivity under various rules and classes of rules and spells out the plan of the book. The notions of sufficient conditions for transitivity (quasi-transitivity), Inada-type necessary conditions for transitivity (quasi-transitivity), and Inada-type necessary and sufficient conditions for transitivity (quasi-transitivity) are rigourously defined.

Abstract: Chapter 2

This chapter contains the basic social choice theoretic concepts, definitions, and propositions that will be needed in the rest of the text. These include, among others, notion of a binary relation, important properties of binary relations, notions of social decision rule and social welfare function, important conditions on social decision rules, Arrow and Gibbard theorems, and the notions of weak Latin Square and Latin Square.

Abstract: Chapter 3

This Chapter is concerned with the Inada-type necessary and sufficient conditions for transitivity and quasi-transitivity under the method of majority decision (MMD). The following theorems are stated and proved in the Chapter.

Let $\mathcal{D}$ be a set of of orderings of the set of social alternatives S.
(i) If the number of individuals n is even and greater than or equal to two then every profile of individual orderings belonging to $\mathcal{D}^n$ yields transitive social weak preference relation under the MMD iff $\mathcal{D}$ satisfies the condition of extremal restriction.
(ii) If the number of individuals n is odd and greater than or equal to three then every profile of individual orderings belonging to $\mathcal{D}^n$ yields transitive social weak preference relation under the MMD iff $\mathcal{D}$ satisfies the condition of weak Latin Square partial agreement.
(iii) If the number of individuals n is greater than or equal to five then every profile of individual orderings belonging to $\mathcal{D}^n$ yields quasi-transitive social weak preference relation under the MMD iff $\mathcal{D}$ satisfies the condition of Latin Square partial agreement.
(iv) If the number of individuals n is four then every profile of individual orderings belonging to $\mathcal{D}^4$ yields quasi-transitive social weak preference relation under the MMD iff $\mathcal{D}$ satisfies the condition of weak extremal restriction.
(v) If the number of individuals n is three then every profile of individual orderings belonging to $\mathcal{D}^3$ yields quasi-transitive social weak preference relation under the MMD iff $\mathcal{D}$ satisfies the condition of Latin Square linear ordering restriction.

The logical relationships among these five conditions are given by: extremal restriction implies weak Latin Square partial agreement; weak Latin Square partial agreement implies Latin Square partial agreement; Latin Square partial agreement implies weak extremal restriction; and weak extremal restriction implies Latin Square linear ordering restriction.

Extremal restriction requires that if a set of orderings of a triple contains a linear ordering of the triple then in any ordering belonging to the set which is of the same Latin Square as the one associated with the linear ordering, the alternative which is the best in the linear ordering must be considered to be at least as good as the alternative which is the worst in the linear ordering. Weak Latin Square partial agreement requires fulfilment of what is required by extremal restriction if a weak Latin Square involving a linear ordering exists. Latin Square partial agreement requires fulfilment of what is required by extremal restriction in case a Latin Square involving a linear ordering exists. Weak extremal restriction requires that in case there is a linear ordering of the triple in question, then it must not be the case that there is an ordering of the triple in which the worst alternative of the linear ordering is uniquely best, there is an ordering of the triple in which the best alternative of the linear ordering is uniquely worst, and both these orderings belong to the same Latin Square which is associated with the linear ordering. Latin Square linear ordering restriction merely requires that there be no Latin Square involving more than one linear ordering.

The literature on the conditions for transitivity and quasi-transitivity under the method of majority decision is vast. The relationships between the conditions of this chapter and conditions discussed in the literature are spelt out in detail in the Chapter.

Abstract: Chapter 4

This Chapter provides a characterization of strict majority rule and Inada-type necessary and sufficient conditions for transitivity and quasi-transitivity under the strict majority rule.

Under the strict majority rule, also called non-minority rule, an alternative x is considered to be socially at least as good as some other alternative y iff a majority of all individuals do not prefer y to x. Thus, under the strict majority rule, between two alternatives social strict preference prevails iff one of the alternatives of the pair is strictly preferred by a majority of all individuals over the other alternative of the pair; otherwise the social indifference prevails. The strict majority rule is characterized by the following seven conditions: (i) independence of irrelevant alternatives (ii) neutrality (iii) monotonicity (iv) weak Pareto-criterion (v) anonymity (vi) a set of individuals is a semidecisive set iff it is a strictly semidecisive set and (vii) every proper superset of a semidecisive set is a decisive set. A set of individuals constitutes a decisive set iff whenever all the individuals in the set consider an alternative to be better than another alternative, then the former is socially preferred to the latter; a set of individuals constitutes a semidecisive set iff whenever all the individuals in the set consider an alternative to be better than another alternative, then the former is socially at least as good as the latter; and a set of individuals constitutes a strictly semidecisive set iff whenever all the individuals in the set consider an alternative to be at least as good as as another alternative, then the former is socially at least as good as the latter.

Let $\mathcal{D}$ be a set of of orderings of the set of alternatives S. It is shown in the Chapter that the strict majority rule defined for an even number of individuals gives rise to transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ satisfies the condition of Latin Square extremal value restriction; and that the strict majority rule defined for an odd number of individuals, number being greater than or equal to three, gives rise to transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ satisfies the condition of weak Latin Square extremal value restriction. A set of orderings $\mathcal{D}$ satisfies Latin Square extremal value restriction over a triple of alternatives A iff there do not exist distinct $a, b, c \in A$ and $R^s, R^t \in \mathcal{D}|A$ such that (i) R^s and R^t belong to the same Latin Square; (ii) alternative a is uniquely best in R^s, and medium in R^t without being worst; and (iii) alternative b is uniquely worst in R^t, and medium in R^s without being best. Latin Square extremal value restriction is satisfied by a set of orderings iff it is satisfied over every triple of alternatives. A set of orderings $\mathcal{D}$ satisfies weak Latin Square extremal value restriction over a triple of alternatives A iff there do not exist distinct $a, b, c \in A$ and $R^s, R^t, R^u \in \mathcal{D}|A$ such that (i) R^s, R^t, R^u form a weak Latin Square; (ii) alternative a is uniquely best in R^s, and medium in R^t without being worst; and (iii) alternative b is uniquely worst in R^t, and medium in R^s without being best. Weak Latin Square extremal value restriction is satisfied by a set of orderings iff it is satisfied over every triple of alternatives. Weak Latin Square extremal value restriction requires what is required by Latin Square extremal value restriction only when a weak Latin Square is present. Thus the transitivity condition for the case of odd number of individuals is less stringent than the transitivity condition for the case of even number of individuals.

When the number of individuals is one or two then the strict majority rule coincides with the weak Pareto-rule. Under the weak Pareto-rule an alternative is socially preferred to another alternative iff everyone prefers the former to the latter. Between two alternatives social indifference prevails if neither of the two alternatives is unanimously preferred over the other. The weak

Pareto-rule yields quasi-transitive social weak preference relation for every profile of individual orderings. Given a set of orderings $\mathcal{D}$ of social alternatives, when the number of individuals is greater than or equal to three, the strict majority rule yields quasi-transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ satisfies the condition of Latin Square unique value restriction. A set of orderings $\mathcal{D}$ satisfies Latin Square unique value restriction over a triple of alternatives A iff there do not exist distinct $a, b, c \in A$ and $R^s, R^t, R^u \in \mathcal{D}|A$ such that (i) alternative b is uniquely medium in R^s, uniquely best in R^t, uniquely worst in R^u, and (ii) R^s, R^t, R^u form a Latin Square. Latin Square unique value restriction is satisfied by a set of orderings iff it is satisfied over every triple of alternatives.

The Chapter also contains a brief survey of the literature on the strict majority rule.

Abstract: Chapter 5

This Chapter is concerned with the derivation of maximally sufficient conditions for transitivity and quasi-transitivity under the class of semi-strict majority rules.

For an alternative x to be socially better than some other alternative y, under the method of majority decision only a majority of nonindifferent individuals are required to have x preferred over y; but under the strict majority rule a majority of all individuals must have x preferred over y for x to be socially better than y. The class of semi-strict majority rules falls in between these two extremes. Under p-semi-strict majority rule, where $0 < p < 1$, for x to be socially better than y the number of individuals preferring x over y must be greater than half of [p(number of individuals nonindifferent between x and y) + $(1-p)$ (number of all individuals)]. When p is close to 1 then the p-semi-strict majority rule is close to the simple majority rule and when p is close to 0 then the p-semi-strict majority rule is close to the strict majority rule.

In the context of transitivity under the class of semi-strict majority rules three conditions on preferences turn out to be relevant. These are: strict placement restriction, partial agreement, and strongly antagonistic preferences (1). A set of orderings of a triple of alternatives satisfies the condition of strict placement restriction iff there is an alternative in the triple such that it is uniquely best in every concerned ordering; or there is an alternative such that it is uniquely worst in every concerned ordering; or there is an alternative such that it is uniquely medium in every concerned ordering;

or there are two distinct alternatives such that in every ordering indifference holds between them. A set of orderings of a triple of alternatives satisfies the condition of partial agreement iff all orderings are concerned and there is an alternative such that it is best in every ordering; or all orderings are concerned and there is an alternative such that it is worst in every ordering. A set of orderings of a triple of alternatives satisfies the condition of strongly antagonistic preferences (1) iff it is a subset of $\{xPyPz, zPyPx, yPxIz\}$ or of $\{xPyPz, zPyPx, xIzPy\}$ for some distinct x, y, z belonging to the triple. Let $\mathcal{D}$ be a set of orderings of the set of alternatives. It is shown in the Chapter that if $\mathcal{D}$ is such that over every triple of alternatives at least one of three conditions of strict placement restriction, partial agreement, and strongly antagonistic preferences (1) holds then p-semi-strict majority rule yields transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$. It is also shown that for every p there exists an n such that p-semi-strict majority rule yields transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ is such that over every triple of alternatives at least one of three conditions of strict placement restriction, partial agreement, and strongly antagonistic preferences (1) holds. Thus it follows that the satisfaction of at least one of the three conditions of strict placement restriction, partial agreement, and strongly antagonistic preferences (1) over every triple of alternatives is maximally sufficient for transitivity under every semi-strict majority rule.

There are two conditions which turn out to be sufficient for quasi-transitivity under every semi-strict majority rule, namely, value-restriction (2) and absence of unique extremal value. The satisfaction of value restriction (2) by a set of orderings of a triple requires that there be an alternative in the triple such that it is not best in any concerned ordering of the triple, or that it is not medium in any concerned ordering of the triple, or that it is not worst in any concerned ordering of the triple. The satisfaction of the condition of absence of unique extremal value by a set of orderings of a triple requires that there be no alternative such that it is uniquely best in some ordering of the set or that there be no alternative such that it is uniquely worst in some ordering of the set. Also, for every $p \in (0, 1)$, there are infinitely many values of n such that p-semi-strict majority rule yields quasi-transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ is such that over every triple of alternatives at least one of the two conditions of value-restriction (2) and absence of unique extremal value holds. Consequently it follows that for every p-semi-strict majority rule, satisfaction of at least one of the two conditions of value-restriction (2) and absence of unique extremal value over every triple of alternatives is maximally sufficient for quasi-transitivity.

Abstract: Chapter 6

This Chapter is concerned with the derivation of conditions for transitivity and quasi-transitivity under the class of special majority rules.

The simple majority rule is a member of the class of majority rules. Let $\frac{1}{2} \leq p < 1$. Under p-majority rule an alternative x is socially at least as good as another alternative y iff the number of individuals preferring y over x is less than or equal to p multiplied by the number of individuals nonindifferent between x and y. Thus, under p-majority rule, for an alternative x to be socially better than some other alternative y, the number of individuals preferring x over y must be greater than p fraction of the number of individuals nonindifferent between x and y. If $p = \frac{1}{2}$, then the rule is the simple majority rule; and if $p > \frac{1}{2}$, then the rule is a special majority rule. The most important special majority rule is the two-thirds majority rule.

In the context of transitivity under the class of special majority rules a condition called the placement restriction turns out to be relevant. A set of orderings of a triple of alternatives satisfies the condition of placement restriction iff there is an alternative in the triple such that it is best in every ordering; or there is an alternative such that it is worst in every ordering; or there is an alternative such that it is uniquely medium in every concerned ordering; or there are two distinct alternatives such that in every ordering indifference holds between them. Let $\mathcal{D}$ be a set of orderings of the set of social alternatives. It is shown in the Chapter that if $\mathcal{D}$ is such that over every triple of alternatives the condition of placement restriction holds then p-majority rule, $\frac{1}{2} < p < 1$, yields transitive social weak preference relation

for every profile of individual orderings belonging to $\mathcal{D}^n$. It is also shown that for every $p \in (\frac{1}{2}, 1)$ there exists an n such that p-majority rule yields transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ is such that it satisfies the condition of placement restriction over every triple of alternatives. This implies that the satisfaction of placement restriction over every triple of alternatives is maximally sufficient for transitivity under every p-majority rule, $\frac{1}{2} < p < 1$.

For the simple majority rule we have the result that if the number of individuals is greater than or equal to five then every profile of individual orderings belonging to $\mathcal{D}^n$ yields quasi-transitive social weak preference relation iff $\mathcal{D}$ is such that it satisfies the condition of Latin Square partial agreement over every triple of alternatives. A similar result holds for every special majority rule. It is shown in this Chapter that for every $p \in (\frac{1}{2}, 1)$ there exists an n' such that if the number of individuals n is greater than or equal to n' then p-majority rule yields quasi-transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ is such that over every triple of alternatives the condition of Latin Square partial agreement holds.

For the two-thirds majority rule more complete results are derived. It is shown that if the number of individuals is greater than or equal to ten then: (i) The two-thirds majority rule yields transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ is such that over every triple of alternatives the condition of placement restriction holds; and (ii) The two-thirds majority rule yields quasi-transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ is such that over every triple of alternatives the condition of Latin Square partial agreement holds.

Abstract: Chapter 7

This Chapter provides a characterization of the class of strict majority rules and derives Inada-type necessary and sufficient conditions for transitivity and quasi-transitivity under the rules belonging to the class.

The strict majority rule discussed in Chapter 4 belongs to the class of strict majority rules. Let $\frac{1}{2} \leq p < 1$. Under p-strict majority rule, an alternative x is considered to be socially at least as good as some other alternative y iff more than p fraction of total number of individuals do not prefer y to x. Thus, under the p-strict majority rule, between two alternatives social strict preference prevails iff one of the alternatives of the pair is strictly preferred by more than p fraction of the total number of individuals over the other alternative of the pair; otherwise the social indifference prevails. If p is $\frac{1}{2}$ then the rule is the strict majority rule discussed in Chapter 4; if $\frac{1}{2} < p < 1$ then the rule will be called a strict special majority rule. The class of strict majority rules is characterized by the following six conditions: (i) independence of irrelevant alternatives (ii) neutrality (iii) monotonicity (iv) weak Pareto-criterion (v) anonymity (vi) a set of individuals is semidecisive iff it is strictly semidecisive. In Chapter 4 it was shown that a social decision rule is the strict majority rule iff it satisfies these six conditions and has the property that every proper superset of a semidecisive set is a decisive set. Thus this last property separates out the strict majority rule from other rules belonging to the class of strict majority rules.

For the purpose of derivation of conditions for transitivity under strict majority rules, the set of rules defined for all possible number of individuals

$\#N = n \geq 2$, N being the set of individuals, is partitioned into two subsets: (i) The rules which are such that there exists a partition (N_1, N_2) of N such that $\#N_1 \leq pn \wedge \#N_2 \leq pn$; (ii) The rules which are such that there does not exist a partition (N_1, N_2) of N such that $\#N_1 \leq pn \wedge \#N_2 \leq pn$. Let $\mathcal{D}$ be a set of of orderings of the set of alternatives S. It is shown in the Chapter that: (i) If the p-strict majority rule is such that there exists a partition (N_1, N_2) of N such that $\#N_1 \leq pn \wedge \#N_2 \leq pn$, then the rule yields transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ satisfies the condition of Latin Square extremal value restriction over every triple of alternatives; and (ii) If the p-strict majority rule is such that there does not exist a partition (N_1, N_2) of N such that $\#N_1 \leq pn \wedge \#N_2 \leq pn$, then the rule yields transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ satisfies the condition of weak Latin Square extremal value restriction over every triple of alternatives.

If the number of individuals $n = \lfloor pn \rfloor + 1$ then the p-strict majority rule coincides with the weak Pareto-rule. The weak Pareto-rule yields quasi-transitive social weak preference relation for every profile of individual orderings. Therefore when $n = \lfloor pn \rfloor + 1$, p-strict majority rule yields quasi-transitive social weak preference relation for every profile of individual orderings. If $n > \lfloor pn \rfloor + 1$ then under every p-strict majority rule every profile of individual orderings belonging to $\mathcal{D}^n$ yields quasi-transitive social weak preference relation iff $\mathcal{D}$ satisfies the condition of Latin Square unique value restriction over every triple of alternatives.

Abstract: Chapter 8

This Chapter is concerned with conditions for transitivity and quasi-transitivity under the class of Pareto-inclusive strict majority rules.

A major drawback of strict majority rules is that they fail to satisfy the Pareto-criterion. The class of Pareto-inclusive strict majority rules, while retaining the 'non-minority' aspect of strict majority rules, by explicitly incorporating the Pareto-criterion remedies this problematic aspect of strict majority rules. Under a Pareto-inclusive p-strict majority rule an alternative x is better than another alternative y iff more than p fraction ($\frac{1}{2} \leq p < 1$) of the total number of individuals prefer x to y or x is Pareto-superior to y.

Let $\mathcal{D}$ be a set of orderings of the set of social alternatives. When the number of individuals is two, Pareto-inclusive p-strict majority rule, $\frac{1}{2} \leq p < 1$, becomes identical with the method of majority decision. Therefore, in view of results of Chapter 3, it follows that when the number of individuals is two, for every Pareto-inclusive p-strict majority rule, extremal restriction completely characterizes all $\mathcal{D}$ such that every profile of individual orderings belonging to $\mathcal{D}^n$ gives rise to transitive social weak preference relation.

If the smallest integer greater than pn is n, the total number of individuals, then the Pareto-inclusive p-strict majority rule coincides with the Pareto-rule. It is established in the Chapter that under the Pareto-rule every profile of individual orderings belonging to $\mathcal{D}^n$ gives rise to transitive social weak preference relation iff $\mathcal{D}$ satisfies over every triple of alternatives at least one of the three conditions of placement restriction, absence of unique

extremal value, and strongly antagonistic preferences (2). A set of orderings of a triple of alternatives satisfies the condition of strongly antagonistic preferences (2) iff it is a subset of $\{xPyPz, zPyPx, yPxIz, xIyIz\}$ or of $\{xPyPz, zPyPx, xIzPy, xIyIz\}$ for some distinct x, y, z belonging to the triple.

It turns out that for every Pareto-inclusive p-strict majority rule satisfaction over every triple of alternatives of placement restriction or absence of unique extremal value by $\mathcal{D}$ is sufficient to ensure transitivity of social weak preference relation generated by every profile belonging to $\mathcal{D}^n$, regardless of the value of n. Furthermore, if n is greater than the smallest integer greater than pn and is such that there exists a three-fold partition of the set of individuals N such that union of no two of them has more than pn individuals then every profile of individual orderings belonging to $\mathcal{D}^n$ gives rise to transitive social weak preference relation iff $\mathcal{D}$ satisfies over every triple of alternatives placement restriction or absence of unique extremal value. Therefore, it follows that for the class of Pareto-inclusive strict majority rules satisfaction over every triple of alternatives of placement restriction or absence of unique extremal value is maximally sufficient for transitivity.

The Chapter provides a complete solution to the problem of quasi-transitivity under the class of Pareto-inclusive strict majority rules. It is shown that: (i) The rules belonging to the subclass of rules which are such that the smallest integer greater than pn is n yield quasi-transitive social weak preference relation for every logically possible profile of individual orderings. (ii) The rules belonging to the subclass of rules which are such that the smallest integer greater than pn is $n-1$ yield quasi-transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ satisfies Latin Square unique value restriction. (iii) The rules belonging to the subclass of rules which are such that the smallest integer greater than pn is less than $n-1$ yield quasi-transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ satisfies over every triple of alternatives Latin Square unique value restriction or limited agreement. A set of orderings of a triple of alternatives satisfies limited agreement iff there exist distinct x, y belonging to the triple such that x is at least as good as y in every ordering of the set.

Abstract: Chapter 9

This Chapter provides a characterization for the class of social decision rules that are simple games as well as for the subclass that are strong simple games; and derives Inada-type necessary and sufficient conditions for transitivity and quasi-transitivity under the rules belonging to the class.

A simple game social decision rule is defined by the condition that under it an alternative x is socially preferred to another alternative y iff all individuals belonging to some decisive set unanimously prefer x to y. It is shown in the Chapter that a social decision rule is a simple game iff it satisfies the following four properties: (i) independence of irrelevant alternatives (ii) neutrality (iii) monotonicity (iv) a set of individuals is a semidecisive set iff it is a strictly semidecisive set. These four properties, along with the two other conditions, namely anonymity and weak Pareto-criterion, figured in the characterization of the class of strict majority rules. Thus the additional conditions of anonymity and weak Pareto-criterion separate the set of strict majority rules from other simple game social decision rules. An important subclass of the simple game social decision rules is that of strong simple game social decision rules. A simple game is a strong simple game iff it is the case that whenever a subset of individuals is not decisive, its complement is decisive. It is shown that a social decision rule is a strong simple game iff it satisfies the properties of (i) independence of irrelevant alternatives (ii) neutrality (iii) monotonicity (iv) the three classes, the class of semidecisive sets, the class of strictly semidecisive sets, and the class of decisive sets, being identical to each other.

For discussing transitivity under simple game social decision rules the set of all simple game social decision rules is partitioned into three subsets: (i) The rules that are null or dictatorial. (ii) The rules that are non-null non-strong simple games (iii) The rules that are non-dictatorial strong simple games. Let $\mathcal{D}$ be a set of of orderings of the set of alternatives S. it is shown that: (i) A simple game social decision rule yields transitive social weak preference relation for every profile of individual orderings iff it is null or dictatorial; (ii) A non-null non-strong simple game social decision rule yields transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ satisfies the condition of Latin Square extremal value restriction over every triple of alternatives; (iii) A non-dictatorial strong simple game social decision rule yields transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ satisfies the condition of weak Latin Square extremal value restriction over every triple of alternatives.

For the purpose of discussing quasi-transitivity under simple game social decision rules the set of all simple game social decision rules is partitioned into two subsets: (i) The simple game social decision rules that are null or are such that there is a unique minimal decisive set (i.e. are oligarchic); (ii) The simple game social decision rules that are non-null and are such that there are at least two minimal decisive sets (i.e. are non-oligarchic). It is shown that: (i) A simple game social decision rule yields quasi-transitive social weak preference relation for every profile of individual orderings iff it is null or is such that there is a unique minimal decisive set; (ii) A non-null simple game social decision rule such that there are at least two minimal decisive sets yields quasi-transitive social weak preference relation for every profile of individual orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ satisfies the condition of Latin Square unique value restriction over every triple of alternatives.

Abstract: Chapter 10

This Chapter is concerned with the class of neutral and monotonic binary social decision rules.

It is shown in this Chapter that a binary social decision rule satisfying the condition of Pareto-indifference and whose domain consists of all logically possible profiles of individual orderings is neutral and monotonic iff Pareto quasi-transitivity holds. Pareto quasi-transitivity requires that for any social alternatives x, y, z; if x is socially better than y, and y is Pareto-superior to z, then x must be socially better than z; and if x is Pareto-superior to y, and y is socially better than z, then x must be socially better than z. As conjunction of Pareto-criterion and quasi-transitivity implies Pareto quasi-transitivity, it follows that every binary social decision rule satisfying the Pareto-criterion and with domain consisting of all logically possible profiles of individual orderings which yields transitive or quasi-transitive social weak preference relation for every profile of individual orderings is neutral and monotonic.

It is shown in the Chapter that for every binary neutral and monotonic social decision rule the condition of strict placement restriction is sufficient to ensure transitivity. That is to say, if $\mathcal{D}$ is a set of orderings of the set of alternatives, then under every binary neutral and monotonic social decision rule the social weak preference relation is transitive for every profile of individual orderings belonging to $\mathcal{D}^n$ if $\mathcal{D}$ satisfies the condition of strict placement restriction over every triple of alternatives. For the class of binary neutral and monotonic social decision rules no condition weaker than this exists that

is sufficient for transitivity under every rule belonging to the class. In other words, the condition of strict placement restriction is maximally sufficient for transitivity for the class of binary neutral and monotonic social decision rules. If we consider the subclass of binary neutral and monotonic social decision rules satisfying the Pareto-criterion then the condition of placement restriction, less stringent than the condition of strict placement restriction, turns out to be maximally sufficient for transitivity.

The Chapter also provides much simpler proofs of the two stanard results regarding quasi-transitivity under the class of binary neutral and monotonic social decision rules, namely, that (i) under every binary neutral and monotonic social decision rule the social weak preference relation is quasi-transitive for every profile of individual orderings belonging to $\mathcal{D}^n$ if $\mathcal{D}$ satisfies the condition of value restriction (2) over every triple of alternatives; and (ii) under every binary neutral and monotonic social decision rule satisfying the Pareto-criterion the social weak preference relation is quasi-transitive for every profile of individual orderings belonging to $\mathcal{D}^n$ if $\mathcal{D}$ satisfies the condition of value restriction (2) or limited agreement over every triple of alternatives.

For the class of binary neutral and monotonic social decision rules the statements of conditions for transitivity and quasi-transitivity become extremely simple when individual orderings are linear. For quasi-transitivity we have the following: (i) Given that the domain consists of all logically possible profiles of individual linear orderings, a binary social decision rule satisfying neutrality and monotonicity yields quasi-transitive social weak preference relation for every profile of individual linear orderings iff there is at most one minimal decisive set. (ii) A binary social decision rule satisfying neutrality and monotonicity which is such that there are at least two minimal decisive sets yields quasi-transitive social weak preference relation for every profile of individual linear orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ is such that it does not contain a Latin Square over any triple of alternatives. For transitivity we have the following: (i) Given that the domain consists of all logically possible profiles of individual linear orderings, a binary social decision rule satisfying neutrality and monotonicity yields transitive social weak preference relation for every profile of individual linear orderings iff it is null or dictatorial. (ii) A non-null non-dictatorial binary social decision rule satisfying neutrality and monotonicity, which is such that for every partition (N_1, N_2) of N either N_1 or N_2 is a decisive set, yields transitive social weak preference relation for every profile of individual linear orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ is such that it does not contain a Latin Square over any triple of alternatives. (iii) A non-null non-dictatorial binary social decision rule satisfying neutrality and monotonicity, which is such that there exists a partition (N_1, N_2) of N such that neither N_1 nor N_2 is a decisive set, yields transitive social weak preference relation for every profile of individual linear orderings belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ is such that it does not contain more than one linear ordering belonging to either of the two Latin Squares over any triple of alternatives.

Abstract: Chapter 11

This Chapter is concerned with the class of neutral and monotonic binary social decision rules and some of its subclasses when individual weak preference relations are reflexive, connected and quasi-transitive rather than orderings.

It is shown in the Chapter that a binary social decision rule with domain consisting of all logically possible profiles of individual reflexive, connected and quasi-transitive weak preference relations is neutral and monotonic iff it satisfies weak Pareto quasi-transitivity. Weak Pareto quasi-transitivity requires that for any social alternatives x, y, z; if x is socially better than y, and y is unanimously preferred to z by every individual, then x must be socially better than z; and if x is is unanimously preferred to y by every individual, and y is socially better than z, then x must be socially better than z.

When individual weak preference relations are reflexive, connected and quasi-transitive, the following propositions hold for quasi-transitivity under the method of majority decision, the class of special majority rules, and the class of social decision rules that are simple games.

Let $\mathcal{D}$ be a set of reflexive, connected, and quasi-transitive relations on the set of social alternatives. For the method of majority decision we have: (i) If the number of individuals is at least five then every profile belonging to $\mathcal{D}^n$ yields quasi-transitive social weak preference relation iff $\mathcal{D}$ is such that it does not contain a Latin Square involving an intransitive relation and it satisfies the condition of Latin Square partial agreement. (ii) If the number of individuals is four then every profile belonging to $\mathcal{D}^4$ yields quasi-transitive social weak preference relation iff $\mathcal{D}$ satisfies the conjunction of weak extremal restriction and the absence of Latin Squares involving intransitive relations. (iii) If the number of individuals is three then every profile belonging to $\mathcal{D}^3$ yields quasi-transitive social preference relation iff $\mathcal{D}$ satisfies the conjunction of Latin Square linear ordering restriction and the absence of Latin Squares involving intransitive relations. (iv) If the number of individuals is two then every profile belonging to $\mathcal{D}^2$ yields quasi-transitive social weak preference relation iff $\mathcal{D}$ satisfies the Latin Square intransitive relation restriction; which requires that there be no Latin Square consisting of two intransitive relations or one intransitive relation and a linear ordering.

For the class of special majority rules we have: (i) The conjunction of Latin Square partial agreement and the absence of Latin Squares involving intransitive relations is sufficient for quasi-transitivity under every special majority rule. (ii) Also, in the case of every special majority rule, if the number of individuals n is sufficiently large then the conjunction of Latin Square partial agreement and the absence of Latin Squares involving intransitive relations completely characterizes the sets $\mathcal{D}$ which are such that every profile belonging to $\mathcal{D}^n$ yields quasi-transitive social weak preference relation.

For the class of social decision rules which are simple games we have: (i) A simple game social decision rule with domain consisting of all logically possible profiles of individual reflexive, connected and quasi-transitive weak preference relations yields quasi-transitive social weak preference relation for every profile iff it is null or there is a unique minimal decisive set. (ii) A non-null simple game social decision rule which is such that there are at least two minimal decisive sets yields quasi-transitive social weak preference relation for every profile belonging to $\mathcal{D}^n$ iff $\mathcal{D}$ satisfies the condition of Latin Square unique value restriction, suitably redefined for the case of reflexive, connected and quasi-transitive weak preference relations.

Abstract: Chapter 12

This chapter contains a concise statement of the main results obtained in the monograph; and lists some of the important open problems.